Negative– and positive–phase–velocity propagation in an isotropic chiral medium moving at constant velocity

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Abstract

Analysis of electromagnetic planewave propagation in a medium which is a spatiotemporally homogeneous, temporally nonlocal, isotropic, chiral medium in a co–moving frame of reference shows that the medium is both spatially and temporally nonlocal with respect to all non–co–moving inertial frames of reference. Using the Lorentz transformations of electric and magnetic fields, we show that plane waves which have positive phase velocity in the co–moving frame of reference can have negative phase velocity in certain non–co–moving frames of reference. Similarly, plane waves which have negative phase velocity in the co–moving frame can have positive phase velocity in certain non–co–moving frames.

Keywords: *Isotropic chiral medium, Lorentz transformation, negative phase velocity, nonlocality*

1   Introduction

Analysis of planewave propagation in a frame of reference that is uniformly moving with respect to the medium of propagation can lead to the emergence and understanding of new phenomenons. In this respect, the Minkowski constitutive relations, as widely described in standard books [1, 2], are strictly appropriate to instantaneously responding mediums only [3]. For realistic material mediums, recourse should be taken to the Lorentz transformation of electromagnetic field phasors [4, 5].

In an earlier study on plane waves in a medium that is spatiotemporally homogeneous, temporally nonlocal, isotropic and chiral in a co–moving frame of reference, we reported that planewave propagation with negative phase velocity (NPV) is possible with respect to a non–co–moving frame of reference, even though the medium does not support NPV propagation in the co–moving frame [6]. That study applies strictly only at low translational speeds. In this paper, we demonstrate by means of an analysis based on the Lorentz–transformed electromagnetic fields in the non–co-moving frame, that our conclusion remains qualitatively valid for realistic mediums even at high translational speeds.

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2 Planewave analysis

We consider a spatiotemporally homogeneous, spatially local, temporally nonlocal, isotropic chiral medium, characterized in the frequency domain by the Tellegen constitutive relations [7]

$$\left.\begin{array}{l} \mathbf{D}' = \epsilon_0 \epsilon_r' \mathbf{E}' + i\sqrt{\epsilon_0\mu_0}\xi'\mathbf{H}' \\ \mathbf{B}' = -i\sqrt{\epsilon_0\mu_0}\xi'\mathbf{E}' + \mu_0\mu_r'\mathbf{H}' \end{array}\right\}, \tag{1}$$

in an inertial frame of reference Σ'. The relative permittivity ϵ_r', relative permeability μ_r' and chirality parameter ξ' are complex–valued functions of the angular frequency ω' if the medium is dissipative, and real–valued if it is nondissipative [1, p. 71]; ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively. The electromagnetic field phasors are related by the Maxwell curl postulates as

$$\left.\begin{array}{l} \nabla \times \mathbf{H}' + i\omega'\mathbf{D}' = \mathbf{0} \\ \nabla \times \mathbf{E}' - i\omega'\mathbf{B}' = \mathbf{0} \end{array}\right\}. \tag{2}$$

Our attention is focused on a plane wave, described by the Σ' phasors

$$\left.\begin{array}{l} \mathbf{E}' = \mathbf{E}_0' \exp\left[i\left(\mathbf{k}' \bullet \mathbf{r}' - \omega't'\right)\right] \\ \mathbf{H}' = \mathbf{H}_0' \exp\left[i\left(\mathbf{k}' \bullet \mathbf{r}' - \omega't'\right)\right] \end{array}\right\}, \tag{3}$$

which propagates in the medium characterized by (1), with wavevector $\mathbf{k}' = k'\hat{\mathbf{k}}'$ and wavenumber $k' = k_0 k_r'$. There are four possibilities for the relative wavenumber: $k_r' \in \{k_{r1}', k_{r2}', k_{r3}', k_{r4}'\}$ where

$$\left.\begin{array}{l} k_{r1}' = \sqrt{\epsilon_r'\mu_r'} + \xi' \\ k_{r2}' = \sqrt{\epsilon_r'\mu_r'} - \xi' \\ k_{r3}' = -k_{r1}' \\ k_{r4}' = -k_{r4}' \end{array}\right\}. \tag{4}$$

With respect to frame Σ', the plane wave is assumed to be uniform; i.e., $\hat{\mathbf{k}}' \in \mathbb{R}^3$.

Suppose that the inertial frame Σ' is moving at constant velocity $\mathbf{v} = v\hat{\mathbf{v}}$ relative to another inertial frame Σ. The electromagnetic field phasors in Σ are related to those in Σ' by the Lorentz transformations [4, 5]

$$\left.\begin{array}{l} \mathbf{E} = (\mathbf{E}' \bullet \hat{\mathbf{v}})\,\hat{\mathbf{v}} + \gamma\left[\left(\underline{I} - \hat{\mathbf{v}}\hat{\mathbf{v}}\right) \bullet \mathbf{E}' - \mathbf{v} \times \mathbf{B}'\right] \\[2mm] \mathbf{B} = (\mathbf{B}' \bullet \hat{\mathbf{v}})\,\hat{\mathbf{v}} + \gamma\left[\left(\underline{I} - \hat{\mathbf{v}}\hat{\mathbf{v}}\right) \bullet \mathbf{B}' + \dfrac{\mathbf{v} \times \mathbf{E}'}{c_0^2}\right] \\[2mm] \mathbf{H} = (\mathbf{H}' \bullet \hat{\mathbf{v}})\,\hat{\mathbf{v}} + \gamma\left[\left(\underline{I} - \hat{\mathbf{v}}\hat{\mathbf{v}}\right) \bullet \mathbf{H}' + \mathbf{v} \times \mathbf{D}'\right] \\[2mm] \mathbf{D} = (\mathbf{D}' \bullet \hat{\mathbf{v}})\,\hat{\mathbf{v}} + \gamma\left[\left(\underline{I} - \hat{\mathbf{v}}\hat{\mathbf{v}}\right) \bullet \mathbf{D}' - \dfrac{\mathbf{v} \times \mathbf{H}'}{c_0^2}\right] \end{array}\right\}, \tag{5}$$

where $\underline{I} = \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}$ is the 3×3 identity dyadic, $\gamma = 1/\sqrt{1 - \beta^2}$ and the relative translational speed $\beta = v/c_0$, with $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ being the speed of light in free space. In terms of the Σ phasors, the plane wave is described by

$$\left.\begin{array}{l} \mathbf{E} = \mathbf{E}_0 \exp\left[i\left(\mathbf{k} \bullet \mathbf{r} - \omega t\right)\right] \\ \mathbf{H} = \mathbf{H}_0 \exp\left[i\left(\mathbf{k} \bullet \mathbf{r} - \omega t\right)\right] \end{array}\right\}. \tag{6}$$

The phasor amplitude vectors $\{\mathbf{E}_0, \mathbf{H}_0\}$ and $\{\mathbf{E}_0', \mathbf{H}_0'\}$ are related via the transformations (5), whereas [4]

$$\mathbf{r} = \left[\underline{I} + (\gamma - 1)\,\hat{\mathbf{v}}\hat{\mathbf{v}}\right] \bullet \mathbf{r}' + \gamma\,\mathbf{v}t', \tag{7}$$

$$t = \gamma\left(t' + \frac{\mathbf{v} \bullet \mathbf{r}'}{c_0^2}\right), \tag{8}$$

$$\mathbf{k} = \gamma\left(\mathbf{k}' \bullet \hat{\mathbf{v}} + \frac{\omega'v}{c_0^2}\right)\hat{\mathbf{v}} + \left(\underline{I} - \hat{\mathbf{v}}\hat{\mathbf{v}}\right) \bullet \mathbf{k}', \tag{9}$$

$$\omega = \gamma\left(\omega' + \mathbf{k}' \bullet \mathbf{v}\right). \tag{10}$$

Since $k' \in \mathbb{C}$ for a dissipative medium, we have from (9) that $\mathbf{k} = k_R \hat{\mathbf{k}}_R + i k_I \hat{\mathbf{k}}_I$ with $k_R \in \mathbb{R}$, $k_I \in \mathbb{R}$, $\hat{\mathbf{k}}_R \in \mathbb{R}^3$, and $\hat{\mathbf{k}}_I \in \mathbb{R}^3$, but $\hat{\mathbf{k}}_R \neq \hat{\mathbf{k}}_I$ in general; i.e., the plane wave is generally nonuniform with respect to Σ. Similarly, from (10) we have that $\omega = \omega_R + i\omega_I$ with $\omega_R \in \mathbb{R}$ and $\omega_I \in \mathbb{R}$. Expressing the Σ phasors as

$$\left. \begin{array}{l} \mathbf{E} = \{\mathbf{E}_0 \exp\left[-\left(\mathbf{k}_I \bullet \mathbf{r} - \omega_I t\right)\right]\} \exp\left[i\left(\mathbf{k}_R \bullet \mathbf{r} - \omega_R t\right)\right] \\ \mathbf{H} = \{\mathbf{H}_0 \exp\left[-\left(\mathbf{k}_I \bullet \mathbf{r} - \omega_I t\right)\right]\} \exp\left[i\left(\mathbf{k}_R \bullet \mathbf{r} - \omega_R t\right)\right] \end{array} \right\}, \tag{11}$$

we note that that the periodic propagation of phase is governed by $\mathbf{k}_R$ and ω_R, whereas attenuation or growth of the wave amplitude is governed by $\mathbf{k}_I$ and ω_I. By writing the phasor amplitudes as $\mathbf{E}_0 = \mathbf{E}_{0R} + i\mathbf{E}_{0I}$ and $\mathbf{H}_0 = \mathbf{H}_{0R} + i\mathbf{H}_{0I}$, the corresponding cycle–averaged Poynting vector may be expressed as

$$\begin{aligned} \mathbf{P} \;=\; & \exp\left(-2\mathbf{k}_I \bullet \mathbf{r}\right) \frac{\omega_R}{2\pi} \times \\ & \left[\mathbf{E}_{0R} \times \mathbf{H}_{0R} \int_{t_0}^{t_0 + \frac{2\pi}{\omega_R}} \cos^2\left(\mathbf{k}_R \bullet \mathbf{r} - \omega_R t\right) \exp\left(2\omega_I t\right) \, dt \right. \\ & - \left(\mathbf{E}_{0I} \times \mathbf{H}_{0R} + \mathbf{E}_{0R} \times \mathbf{H}_{0I}\right) \times \\ & \quad \int_{t_0}^{t_0 + \frac{2\pi}{\omega_R}} \cos\left(\mathbf{k}_R \bullet \mathbf{r} - \omega_R t\right) \sin\left(\mathbf{k}_R \bullet \mathbf{r} - \omega_R t\right) \exp\left(2\omega_I t\right) \, dt \\ & \left. + \mathbf{E}_{0I} \times \mathbf{H}_{0I} \int_{t_0}^{t_0 + \frac{2\pi}{\omega_R}} \sin^2\left(\mathbf{k}_R \bullet \mathbf{r} - \omega_R t\right) \exp\left(2\omega_I t\right) \, dt \right] \end{aligned} \tag{12}$$

for a cycle beginning at time $t = t_0$. The phase velocity is given by

$$\mathbf{v}_p = \frac{\omega_R}{k_R} \hat{\mathbf{k}}_R. \tag{13}$$

Whether the plane wave has positive phase velocity (PPV) or negative phase velocity (NPV) in the reference frame Σ is determined by the sign of $\mathbf{v}_p \bullet \mathbf{P}$: positive for PPV and negative for NPV. Criterions for determining whether the phase velocity is positive or negative with respect to the reference frame Σ' are presented elsewhere [8, 9].

3 Numerical results and discussion

For the sake of illustration, let us consider the cycle–averaged Poynting vector evaluated at the point $\mathbf{r} = \mathbf{0}$ with the temporal averaging starting from $t_0 = 0$; i.e.,

$$\begin{aligned} \mathbf{P}|_{\mathbf{r}=\mathbf{0}} \;=\; & \frac{\omega_R}{8\pi\omega_I |\omega|^2} \left[\exp\left(\frac{4\pi\omega_I}{\omega_R}\right) - 1 \right] \left[\left(|\omega|^2 + \omega_I^2\right) \mathbf{E}_{0R} \times \mathbf{H}_{0R} \right. \\ & \left. - \omega_R \omega_I \left(\mathbf{E}_{0I} \times \mathbf{H}_{0R} + \mathbf{E}_{0R} \times \mathbf{H}_{0I}\right) + \omega_R^2 \mathbf{E}_{0I} \times \mathbf{H}_{0I} \right]. \end{aligned} \tag{14}$$

Without loss of generality, let us assume that the plane wave propagates along the z' Cartesian axis; i.e., $\hat{\mathbf{k}}' = \hat{\mathbf{z}}'$. It follows then from the Maxwell curl postulates (2) that $\mathbf{E}'_0$ lies in the $x'y'$ plane with

$$\left. \begin{array}{ll} \hat{\mathbf{y}}' \bullet \mathbf{E}'_0 = i\hat{\mathbf{x}}' \bullet \mathbf{E}'_0 & \text{for} \quad k'_r = k'_{r1}, k'_{r4} \\ \hat{\mathbf{y}}' \bullet \mathbf{E}'_0 = -i\hat{\mathbf{x}}' \bullet \mathbf{E}'_0 & \text{for} \quad k'_r = k'_{r2}, k'_{r3} \end{array} \right\}. \tag{15}$$

Further, we take the velocity $\mathbf{v}$ to lie in the $x'z'$ Cartesian plane as per

$$\hat{\mathbf{v}} = \hat{\mathbf{x}}' \sin\theta + \hat{\mathbf{z}}' \cos\theta. \tag{16}$$

In Figure 1 the distributions of PPV and NPV in the reference frame Σ for the dissipative scenario wherein $\epsilon'_r = 6.5 + 1.5i$, $\xi' = 1 + 0.2i$ and $\mu'_r = 3 + 0.5i$ are displayed for $k'_r \in \{k'_{r1}, k'_{r2}, k'_{r3}, k'_{r4}\}$. Clearly,

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for all four values of k'_r, propagation is of the PPV type for $\beta = 0$ (i.e., with respect to the Σ' frame). As the relative translational speed increases, the phase velocity of the plane waves corresponding to k'_{r1} and k'_{r2} eventually becomes negative provided that $\pi/2 < \theta < \pi$. In contrast the plane waves corresponding to k'_{r3} and k'_{r4} have NPV at sufficiently large values of β provided that $0 < \theta < \pi/2$.

Now, let us look at the scenario where the isotropic chiral medium supports NPV propagation in the co–moving reference frame. This situation arises when $|\mathrm{Re}\,\{\xi'\}| > \mathrm{Re}\,\left\{\sqrt{\epsilon'_r}\sqrt{\mu'_r}\right\}$, for example [8, 9]. We take $\epsilon'_r = 6.5 + 1.5i$, $\xi' = 10 + 2i$ and $\mu'_r = 3 + 0.5i$. The corresponding distributions of NPV and PPV are mapped against β and θ in Figure 2. We see that the plane waves corresponding to relative wavenumbers k'_{r1} and k'_{r3} have NPV when β is small but have PPV when β is sufficiently large. On the other hand, the plane waves corresponding to relative wavenumbers k'_{r2} and k'_{r4} have PPV when β is small but have NPV when β is sufficiently large.

To conclude, by means of the Lorentz–transformed electromagnetic fields, we have demonstrated that a plane wave with PPV in an isotropic chiral medium can have NPV when observed from a no–co–moving inertial reference frame. Similarly a NPV plane wave in the co–moving frame can be PPV from a non–co–moving frame.

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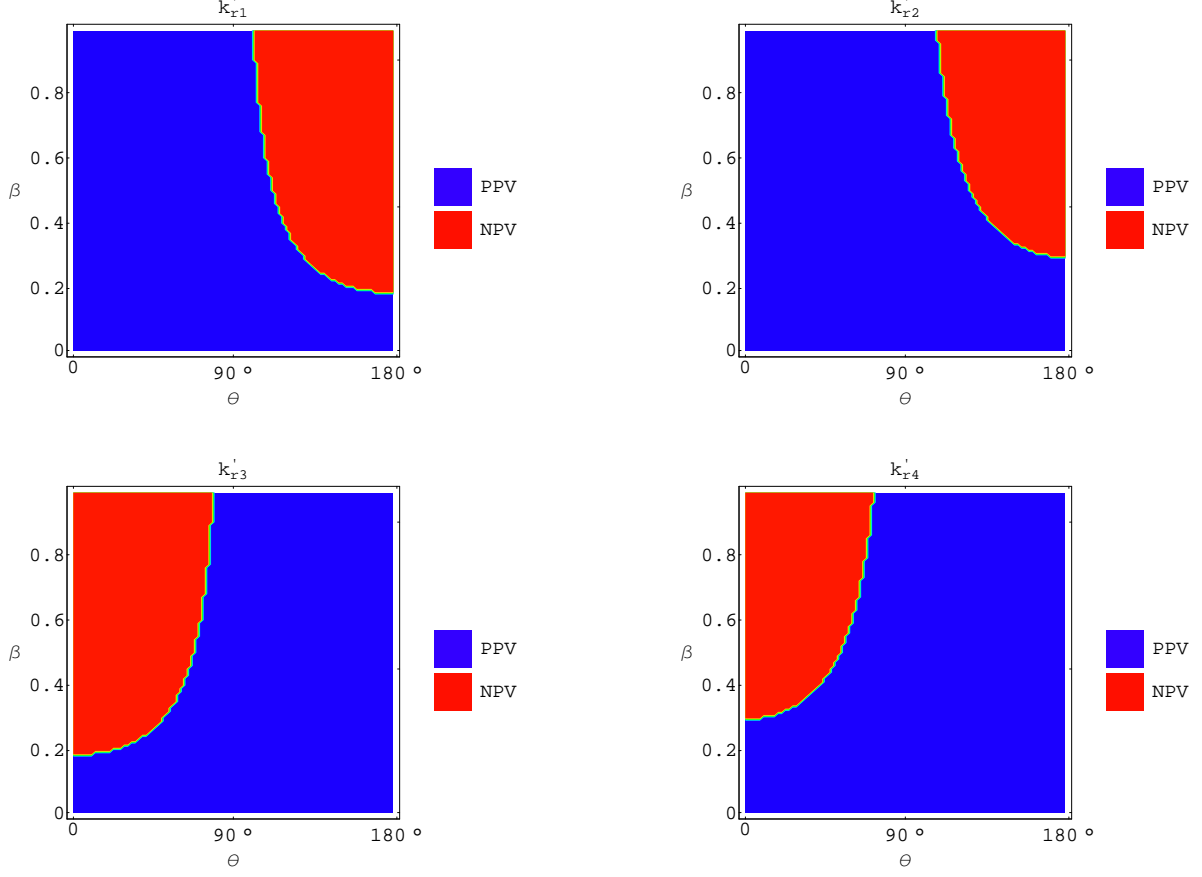
Figure 1: The distribution of positive phase velocity (PPV) and negative phase velocity (NPV) in the Σ reference frame, in relation to $\beta \in [0,1)$ and $\theta \in [0°, 180°)$, for $k'_r \in \{k'_{r1}, k'_{r2}, k'_{r3}, k'_{r4}\}$. Here, $\epsilon'_r = 6.5 + 1.5i$, $\xi' = 1 + 0.2i$ and $\mu'_r = 3 + 0.5i$.
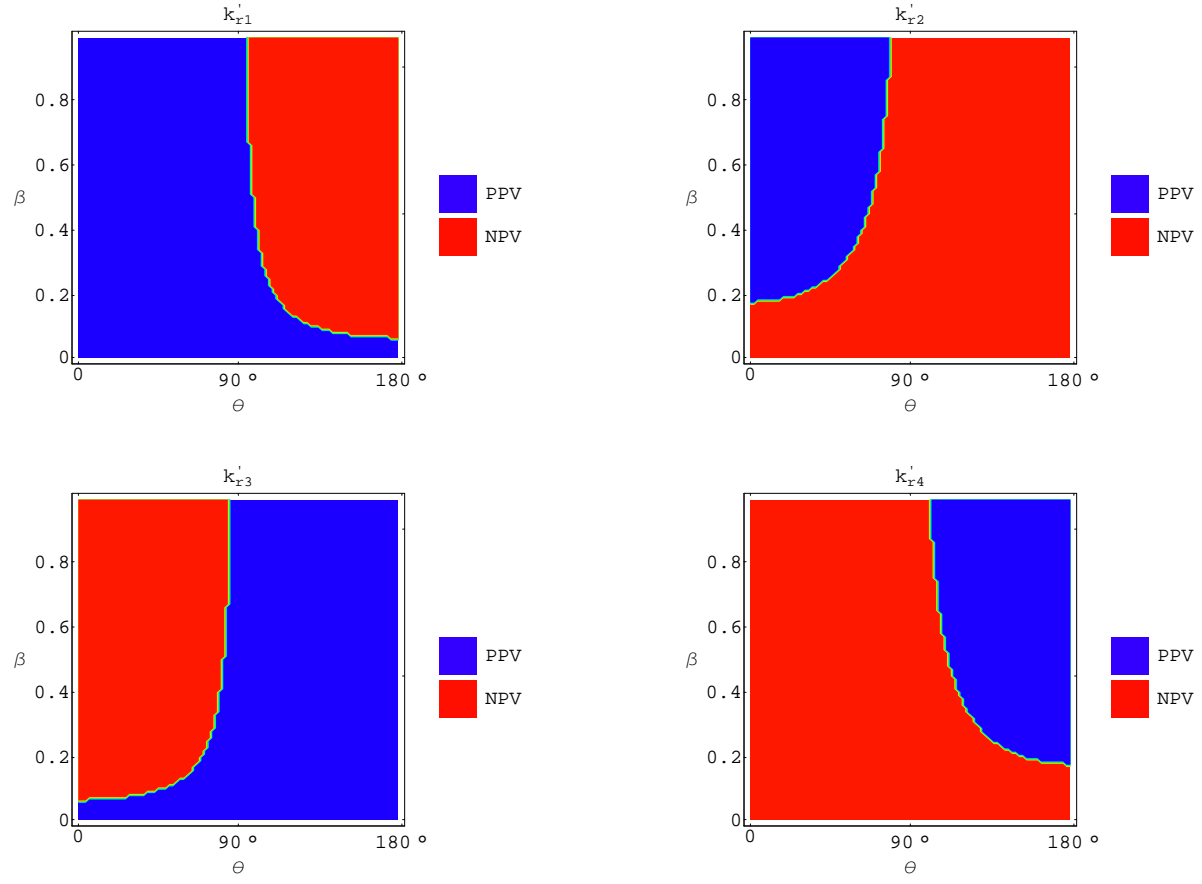
Figure 2: As Figure 1 but with $\xi' = 10 + 2i$.